Section 4

4.1 The following assumptions can be made:

(a) the change in heat energy of the tea is proportional to the change in its temperature and to the mass of the tea (as explained in MST209 *Unit 15*);

(b) all heat losses due to radiation are ignored;

(c) all heat losses due to conduction and convection through the sides and bottom of the cup are ignored;

(d) in the steady state, the rate of heat transfer by convection at a surface between a fluid and another substance is proportional to the difference between the temperature of the surface and the temperature of the substance, and proportional to the area of the surface;

(e) the tea, apart from a thin layer close to its surface with the air, has a uniform temperature;

(f) the surrounding air, apart from a thin layer close to the surface of the tea, has a uniform constant temperature;

(g) over a short time interval, the steady-state model for the rate of heat transfer by convection provides a good approximation to the rate of heat energy loss, and this approximation improves as the time interval becomes shorter;

(h) changes in the heat energy of the material of the cup are ignored;

(i) the surface of the tea within the cup is a disc.

4.2 The variables are: t, $\Theta(t)$, δE, $\Theta_{\text{sur}}(t)$ and $q(t)$. The rest are parameters.

4.3 It is better to have a common symbol for all related variables, and to distinguish them from each other by subscripts.

Using $\Theta_{\text{sur}}(t)$ instead of Θ_{sur} emphasizes that it is a function of t and not a parameter. However, the functional notation may sometimes be omitted, for convenience. (This also applies to other dependent variables.)

The variable name $\Theta_{\text{s}}(t)$ may also be used; it is shorter to write than $\Theta_{\text{sur}}(t)$, but its meaning may not be recognized so easily.

4.4 Rearranging the two equations, the temperature differences are

$$\Theta - \Theta_{\text{sur}} = \frac{q}{h_{\text{tea}}A} \quad \text{and} \quad \Theta_{\text{sur}} - \Theta_{\text{air}} = \frac{q}{h_{\text{air}}A}.$$

In order to eliminate Θ_{sur}, these two equations are added to obtain

$$\Theta - \Theta_{\text{air}} = \frac{q}{A}\left(\frac{1}{h_{\text{tea}}} + \frac{1}{h_{\text{air}}}\right).$$

Hence

$$q = UA(\Theta - \Theta_{\text{air}}), \quad \text{where} \quad U = \left(\frac{1}{h_{\text{tea}}} + \frac{1}{h_{\text{air}}}\right)^{-1}.$$

4.5 Yes, all the assumptions have been used. This is a useful check to make, to ensure that there are no redundant assumptions.

4.6 **(a)** If $\Theta > \Theta_{\text{air}}$, then the temperature of the tea would decrease, that is, $d\Theta/dt < 0$. If $\Theta < \Theta_{\text{air}}$ (e.g. iced tea), then the expectation is that $d\Theta/dt > 0$. If $\Theta = \Theta_{\text{air}}$, then the temperature of the tea should not change, so that $d\Theta/dt = 0$. Furthermore, the magnitude $|d\Theta/dt|$ of the rate of change of temperature would increase as a result of any increase in the magnitude $|\Theta - \Theta_{\text{air}}|$ of the temperature difference.

(b) Equation (4.7) says that the rate of change of temperature, $d\Theta/dt$, is proportional to the temperature difference $\Theta - \Theta_{\text{air}}$, with a *negative* constant of proportionality. So this equation is consistent with all the statements in part (a).

4.7 The differential equation can be solved by using either the integrating factor method or the separation of variables method (both described in MST209 *Unit 2*). Using the integrating factor method, first rewrite the equation in the form

$$\frac{d\Theta}{dt} + \lambda\Theta(t) = \lambda\Theta_{\text{air}}.$$

The integrating factor is $p = \exp(\int \lambda\, dt) = e^{\lambda t}$. Multiplying the differential equation by this factor and then integrating,

$$e^{\lambda t}\Theta(t) = \int \lambda\Theta_{\text{air}} e^{\lambda t}\, dt = \Theta_{\text{air}} e^{\lambda t} + C,$$

where C is an arbitrary constant. Rearranging,

$$\Theta(t) = \Theta_{\text{air}} + Ce^{-\lambda t}.$$

Putting $t = 0$ and using the initial condition $\Theta(0) = \Theta_0$ gives $C = \Theta_0 - \Theta_{\text{air}}$. Hence the required particular solution of the differential equation is

$$\Theta(t) = \Theta_{\text{air}} + (\Theta_0 - \Theta_{\text{air}})e^{-\lambda t}.$$

4.8 Since $\lambda = UA/(mc)$,

$$U = \frac{\lambda mc}{A} = \frac{3.403 \times 10^{-4} \times 0.25 \times 4190}{0.004\,07} = 87.55\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}.$$

Also

$$\frac{1}{U} = \frac{1}{h_{\text{air}}} + \frac{1}{h_{\text{tea}}},$$

so that

$$\frac{1}{h_{\text{air}}} = \frac{1}{U} - \frac{1}{h_{\text{tea}}} = \frac{1}{87.55} - \frac{1}{500} = 0.009\,422.$$

Hence $h_{\text{air}} = 106.1\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$.

Therefore, the experimental value for λ would arise from a U-value for the convective heat transfer from the tea to the air of $87.6\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$, which corresponds to a convective heat transfer coefficient from the surface to the air of $106\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$.

4.9 **(a)** In either case, $h_{\text{air}} = 10$. If $h_{\text{tea}} = 50$, then $U = 8.3\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$, whereas if $h_{\text{tea}} = 1000$ then $U = 9.9\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$.

(b) In either case, $h_{\text{tea}} = 500$. If $h_{\text{air}} = 2$, then $U = 2.0\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$, whereas if $h_{\text{air}} = 25$ then $U = 24\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$.

(c) The value of U is much more sensitive to changes in the value of h_{air} than to changes in the value of h_{tea}, so this will also be the case for the estimate for T.

4.10 A better value for h_{air} might lead to a better estimate for T, the time taken for the tea to cool. In fact, as you saw in the solution to Exercise 4.8, a value of $106\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$ for h_{air}, based on the experimental data, would provide a good estimate for T. However, this value is well outside the given range 2–25 for convective heat transfer coefficients for gases, so the model seems unlikely to be a valid one.

Furthermore, even if there is a better estimate for h_{air} in the range 2–25, the model would still predict an exponential relationship between temperature and time. So the model would still not be satisfactory, as the data show that this relationship is not quite exponential.

4.11 The two most simple shapes that resemble a cup are a cylinder and a hemisphere. For the former, an extra parameter, the height of the cup, is required.

Although either could be pursued, the model to be proposed here will assume a cylindrical shape.

4.12 The value of λ will be larger in the revised model, so that the time predicted for the tea to cool to a drinkable temperature will be less. Hence the revised model is likely to be more accurate than the first model. However, the solution is still of the same form as for the first model and so will not address the qualitative deficiencies of that model.

4.13 With $h_{\text{air}} = 32\,\text{W}\,\text{m}^{-2}\,\text{K}^{-1}$,

$$U = \left(\frac{1}{h_{\text{tea}}} + \frac{1}{h_{\text{air}}}\right)^{-1} = \left(\frac{1}{500} + \frac{1}{32}\right)^{-1} = 30.075$$

and

$$U_{\text{side}} = \left(\frac{1}{h_{\text{tea}}} + \frac{b}{\kappa} + \frac{1}{h_{\text{air}}}\right)^{-1} = \left(\frac{1}{500} + \frac{0.003}{1.5} + \frac{1}{32}\right)^{-1} = 28.369,$$

so that

$$\lambda = \frac{UA + U_{\text{side}}A_{\text{side}}}{mc} = \frac{30.075 \times 0.004\,07 + 28.369 \times 0.0139}{0.25 \times 4190} = 0.000\,494.$$

From Equation (4.9), the time taken for the tea to become drinkable is

$$T = \frac{1}{\lambda}\ln\left(\frac{\Theta_0 - \Theta_{\text{air}}}{\Theta_T - \Theta_{\text{air}}}\right) = 781.$$

Hence the given value of h_{air} leads to a predicted cooling time of about 13 minutes, which is very close to the experimental value.

This value for h_{air} is in the range for forced convection, and so the model could only be used if forced convection was taking place. It is possible that there was forced convection when the experiment was performed, so the model with this value for h_{air} may be satisfactory for finding the cooling time, but it would need a further experiment to test this. However, the model remains unsatisfactory for the purpose of discovering how, qualitatively, the temperature of the tea cools over time.

4.14 Taking radiative heat transfer into account, the differential equation would be of the form

$$\frac{d\Theta}{dt} = -\lambda(\Theta(t) - \Theta_{\text{air}}) - a(\Theta(t)^4 - \Theta_{\text{air}}^4),$$

where a is some parameter and $\Theta(t)$ and Θ_{air} are measured in kelvins. The additional term would have most effect when the difference between $\Theta(t)$ and Θ_{air} is greatest, and a lesser effect as the difference reduces. Including radiative heat transfer, therefore, could help to explain the qualitative difference between the models and the experimental data.

For the smallish temperature difference present in this model, it is good enough to replace the fourth power function by a straight-line approximation. The effect would be to increase the value of λ, and comments similar to those for Solution 4.12 would then apply.

4.15 **(a)** Since convection at the sides of the cup will be included, modify Assumptions (c), (e) and (f). Since the energy in the material of the cup is being taken into account, modify Assumption (h). The new set of assumptions, with the changes highlighted in italics and omissions marked by ..., is as follows:

(a) the change in heat energy of the tea is proportional to the change in its temperature and to the mass of the tea;

(b) all heat losses due to radiation are ignored;

(c) all heat losses due to conduction ... through the sides and bottom of the cup are ignored;

(d) in the steady state, the rate of heat transfer by convection at a surface between a fluid and another substance is proportional to the difference between the temperature of the surface and the temperature of the substance, and proportional to the area of the surface;

(e) the tea, apart from a thin layer close to its surface with the air *and to its surface with the sides of the cup*, has a uniform temperature;

(f) the surrounding air, apart from a thin layer close to the surface of the tea *and to the surface of the sides of the cup*, has a uniform constant temperature;

(g) over a short time interval, the steady-state model for the rate of heat transfer by convection provides a good approximation to the rate of heat energy loss, and this approximation improves as the time interval becomes shorter;

(h) *the change* in the heat energy of the material of the cup *is proportional to the change in its temperature and to the mass of the cup*;

(i) *the cup is cylindrical in shape*;

(j) *the thickness of the cup wall is small when compared to other dimensions, so that the internal and external surface areas of the cup can be taken to be equal.*

(b) The additional variables and parameters are given in the following table.

Symbol	Definition	Units
$\Theta_{\text{cup}}(t)$	the temperature of the cup at time t	°C
m_{cup}	the mass of the cup	kg
c_{cup}	the specific heat of the material from which the cup is made	$\text{J kg}^{-1}\,\text{K}^{-1}$
δE_{cup}	the change in the heat energy of the cup during the small time interval $[t, t+\delta t]$	J
$q_{\text{teacup}}(t)$	the rate of heat transfer from the tea to the cup at time t	W
$q_{\text{cupair}}(t)$	the rate of heat transfer from the cup to the air at time t	W
A_{side}	the surface area of the sides of the cup	m^2

In addition, assume that the values of h_{tea} and h_{air} also apply at the internal and external surfaces of the cup, respectively.

(c) First consider the tea. As in the first model, over a small time interval $[t, t+\delta t]$, the change in energy is

$$\delta E = mc(\Theta(t+\delta t) - \Theta(t)).$$

During this time interval, energy is transferred to the air from the surface of the tea and to the cup through its sides. The rates of heat energy transfer are, respectively,

$$q(t) = UA(\Theta(t) - \Theta_{\text{air}}),$$

where $U = (1/h_{\text{tea}} + 1/h_{\text{air}})^{-1}$, as before, and

$$q_{\text{teacup}}(t) = h_{\text{tea}}A_{\text{side}}(\Theta(t) - \Theta_{\text{cup}}(t)).$$

Since there is no heat energy input to the tea in the interval $[t, t+\delta t]$, the input–output principle, applied to the heat energy of the tea, gives

$$\delta E \simeq 0 - (q(t) + q_{\text{teacup}}(t))\delta t,$$

so that the change in the heat energy of the tea is given by

$$\begin{aligned} mc(\Theta(t+\delta t) - \Theta(t)) \\ \simeq -(UA(\Theta(t) - \Theta_{\text{air}}) \\ + h_{\text{tea}}A_{\text{side}}(\Theta(t) - \Theta_{\text{cup}}(t)))\delta t. \end{aligned}$$

Dividing both sides by $mc\,\delta t$, and then letting δt tend to zero, gives the differential equation

$$\frac{d\Theta}{dt} = -\frac{UA + h_{\text{tea}}A_{\text{side}}}{mc}\Theta(t) + \frac{h_{\text{tea}}A_{\text{side}}}{mc}\Theta_{\text{cup}}(t) + \frac{UA}{mc}\Theta_{\text{air}}. \qquad \text{(S.8)}$$

(This differential equation describes how the temperature of the tea varies with time. It contains the unknown function $\Theta_{\text{cup}}(t)$, the temperature of the cup.)

Now apply the input–output principle to the heat energy in the cup over the time interval $[t, t+\delta t]$. The change in the heat energy of the cup in this time interval is given by

$$\delta E_{\text{cup}} = m_{\text{cup}}c_{\text{cup}}(\Theta_{\text{cup}}(t+\delta t) - \Theta_{\text{cup}}(t)).$$

During this time interval, energy is transferred from the tea to the cup through the inner sides of the cup, and from the cup to the air through the outer sides of the cup. The rates of heat energy transfer are, respectively,

$$q_{\text{teacup}}(t) = h_{\text{tea}} A_{\text{side}}(\Theta(t) - \Theta_{\text{cup}}(t))$$

and

$$q_{\text{cupair}}(t) = h_{\text{air}} A_{\text{side}}(\Theta_{\text{cup}}(t) - \Theta_{\text{air}}).$$

The input–output principle, applied to the heat energy of the cup over the time interval $[t, t + \delta t]$, gives

$$\begin{aligned}\delta E_{\text{cup}} &= \text{input} - \text{output}\\ &\simeq q_{\text{teacup}}(t)\delta t - q_{\text{cupair}}(t)\delta t,\end{aligned}$$

so that the change in the heat energy of the cup is given by

$$\begin{aligned}m_{\text{cup}} c_{\text{cup}}&(\Theta_{\text{cup}}(t + \delta t) - \Theta_{\text{cup}}(t))\\ &\simeq (h_{\text{tea}} A_{\text{side}}(\Theta(t) - \Theta_{\text{cup}}(t))\\ &\quad - h_{\text{air}} A_{\text{side}}(\Theta_{\text{cup}}(t) - \Theta_{\text{air}}))\delta t.\end{aligned}$$

Dividing both sides by $m_{\text{cup}} c_{\text{cup}} \delta t$, and then letting δt tend to zero, gives the differential equation

$$\begin{aligned}\frac{d\Theta_{\text{cup}}}{dt} &= \frac{h_{\text{tea}} A_{\text{side}}}{m_{\text{cup}} c_{\text{cup}}}\Theta(t) - \frac{(h_{\text{tea}} + h_{\text{air}})A_{\text{side}}}{m_{\text{cup}} c_{\text{cup}}}\Theta_{\text{cup}}(t)\\ &\quad + \frac{h_{\text{air}} A_{\text{side}}}{m_{\text{cup}} c_{\text{cup}}}\Theta_{\text{air}}. \qquad \text{(S.9)}\end{aligned}$$

(This differential equation describes how the temperature of the cup varies with time. It contains the unknown function $\Theta(t)$, the temperature of the tea.)

4.16 **(a)** The following two situations for cooling of the tea are to be compared: (i) a constant air temperature $\Theta_{\text{air}} = C$; (ii) an air temperature $\Theta_{\text{air}}(t)$, with average value C over the time interval under consideration, which starts just above C and decreases slowly to finish just below C. The qualitative behaviour of the rate of cooling is likely to be very similar in these two cases.

For a smaller difference between the tea and air temperatures, the rate of cooling (heat loss) of the tea will be less. In the early stages, therefore, the tea temperature in situation (ii) will be slightly higher at a given time than the corresponding tea temperature in situation (i). However, for a given tea temperature in the later stages, the heat loss will be slightly greater in situation (ii). Depending on the precise nature of the function $\Theta_{\text{air}}(t)$ and on the length of the time interval considered, the final tea temperature for situation (ii) may or may not be lower than that for situation (i).

(b) A linear relationship for the ambient temperature in terms of time is

$$\Theta_{\text{air}}(t) = 18 - \frac{t}{90 \times 60} = 18 - 0.000\,185t.$$

(c) Incorporating this into the first revised model gives

$$\frac{d\Theta}{dt} = -\lambda(\Theta(t) - 18 + 0.000\,185t),$$

where

$$\lambda = \frac{UA + U_{\text{side}} A_{\text{side}}}{mc} \simeq 0.000\,166.$$

The integrating factor method gives the solution as

$$\begin{aligned}\Theta(t) &= \left(\Theta_0 - 18 - \frac{1}{\lambda}0.000\,185\right)e^{-\lambda t}\\ &\qquad + 18 + \frac{1}{\lambda}0.000\,185 - 0.000\,185t\\ &\simeq 60.9e^{-0.000\,166t} + 19.1 - 0.000\,185t.\end{aligned}$$